Effective noise in thresholded intensity distribution: influence on centroid statistics

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It is usual to preprocess data before reduction, but it is not so common to study how this operation affects the final results. Determination of the centroid is a relevant task for many optical measurement devices, and the centroid is very often calculated over thresholded data. The influence of preprocessing thresholding algorithms on the statistical properties of intensity data affected by additive Gaussian noise is described as a different effective additive signal perturbation. Theoretical, simulated, and experimental analyses of the model of the effective noise were performed, and good agreement among the analyses was obtained. Direct extension of the analyses from the influence of preprocessing to centroid determination is also presented.

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Preprocessing algorithms are often used to improve the accuracy of centroid calculation.[1] Among them, thresholding is one of the more prominent methods. Thresholding is important because it is a fast operation for reducing undesirable background and random noise. Images of focal spots taken from a Shack–Hartmann wave-front sensor or from optical triangulation devices are usually used in this kind of procedure. However, in the literature not much attention has been paid to how these thresholding algorithms affect the final results (except in studies related to spatial truncation of the recorded irradiance[2]).

In this Letter we begin by showing how additive noise with a Gaussian probability distribution effectively becomes, as a result of thresholding, another kind of perturbation with a noticeably different nature. Theoretical deduction of this probability distribution and its two first characteristic moments is presented, along with numerical and experimental proofs. Taking advantage of this theoretical deduction, we analyze the effect of thresholding on the statistical properties of the centroid of a certain light distribution. Finally, we discuss the conditions in which influence of thresholding cannot be neglected.

Currently CCDs are widely used as photon detectors integrated into imaging devices. In this way, we obtain measurements of the light-intensity distribution, I_j, in discrete samples called pixels. In the case of negligible photon noise, it is usual to model these measurements as being affected by additive Gaussian noise with zero mean:

$$I_j = I_{0j} + n_j, \qquad \langle I_j \rangle = I_{0j}, \tag{1}$$

where the angle brackets denote time averaging, j is the pixel coordinate, $I_{0j}t$ is the noiseless intensity, and n_j is the noise perturbation.

The application of thresholding over the discrete intensity distribution, I_j, is equivalent to a nonlinear transformation of I_j to I_{Uj}, so

$$I_{Uj} = \begin{cases} I_j & \text{if } I_j \geq U_j \\ 0 & \text{if } I_j < U_j \end{cases}, \tag{2}$$

where U_j is the threshold-level distribution and I_{Uj} the thresholded intensity.

But how does thresholding affect the final distribution? We explain this influence as an effective change in statistical noise properties, also assuming an additive model for the noise:

$$I_{Uj} = I_{0j} + N_j. \tag{3}$$

In this model the probability distribution of N_j is not perfectly Gaussian but is clipped. Moreover, the point at which the Gaussian is clipped is directly related to the difference between noiseless intensity value I_{0j} and thresholding level U_j. In this way,

$$P(N_j) = \begin{cases} \dfrac{1}{\sqrt{2}\,\pi\sigma_n} \exp\left(-\dfrac{{N_j}^2}{2{\sigma_n}^2}\right) & \text{if } U_j - I_{0j} \leq n_j \\ \delta(N + I_{0j}) \times \displaystyle\int_{-\infty}^{U_j - I_{0j}} \dfrac{1}{\sqrt{2}\,\pi\sigma_n} \exp\left(-\dfrac{{t_j}^2}{2{\sigma_n}^2}\right) \mathrm{d}t & \text{if } U_j - I_{0j} > n_j \\ 0 & \text{rest of the cases} \end{cases}, \tag{4}$$

where $P(N_j)$ is the probability distribution of N for the j-coordinate pixel.

It is easy to realize the consistency of this proposal. By the definition of thresholding [Eq. (2)], the noisy pixels that cannot overcome the threshold are set to zero. This happens when the original noise value, n, is smaller than the difference $U_j - I_{0j}$. In addition, the new proposed noise, N, has a probability of being the opposite value of noiseless intensity I_0 that is equal to the sum of the probability of each one of the cases that could not overcome the threshold level. In the case of no thresholding ($U_j = -\infty$) the domain of the first condition in Eq. (4) extends to all the space, and we recover the expression for the natural probability distribution of the noise. In practice, we can consider this to be true when $U_j - I_{0j} > 3\sigma_n$.

Next, using proposed probability distribution function (4), we proceed to deduce the first and second statistical moments, which give information on the effective noise behavior. By straightforward integration of all the possible N values we obtain

$$\langle N_j \rangle = \frac{1}{\sqrt{2\pi}}\,\sigma_n \exp\left(-\frac{U_j - I_{0j}}{2{\sigma_n}^2}\right) - \frac{I_{0j}}{2}\left[1 + \mathrm{erf}\left(\frac{U_j - I_{0j}}{\sqrt{2}\,\sigma_n}\right)\right], \quad (5)$$

$$\sigma_{Nj}{}^2 = \frac{{\sigma_n}^2}{2}\left[1 - \mathrm{erf}\left(\frac{U_j - I_{0j}}{\sqrt{2}\,\sigma_n}\right)\right] + \frac{1}{\sqrt{2\pi}}\,\sigma_n(U_j - I_{0j}) \times \exp\left[-\frac{(U_j - I_{0j})^2}{2{\sigma_n}^2}\right] - \frac{1}{2\pi}\,{\sigma_n}^2 \exp\left[-\frac{(U_j - I_{0j})^2}{2{\sigma_n}^2}\right] + {I_{0j}}^2\left[1 + \mathrm{erf}\left(\frac{U_j - I_{0j}}{\sqrt{2}\,\sigma_n}\right)\right]\cdot \quad (6)$$

We can observe from Eqs. (5) and (6) how the mean and variance depend on the difference between I_{0j} and U_j for each pixel. As a limit case, where $U_j \ll I_{0j}$, we have $\mathrm{erf}(-\infty) = -1$, which implies that $\langle N_j \rangle = 0$ and $\sigma_{Nj}{}^2 = \langle N_j{}^2 \rangle - \langle N_j \rangle^2 = {\sigma_n}^2$, as would be expected.

We checked these analytical results by performing a numerical simulation. We generated a statistically significant sample set (10^5 realizations) of random Gaussian values with zero mean and ${\sigma_n}^2$ variance. The first two moments of the sample were repeatedly evaluated for one series of threshold values. In Fig. 1, results are shown as open circles, and the theoretical results are represented by the solid curves. It is shown in Fig. 1(a) how the mean value of the effective noise evolves in relation to the level of the selected threshold, and Fig. 1(b) shows the evolution of the variance's root. We observe a very good agreement between the results of the simulation and the theoretical ones. From this figure we can observe the special behavior of the effective noise and its noticeable difference from the previous model.

To check the consistency of our model we also performed an experimental validation. We recorded two groups of 1500 intensity samples with a covered CCD camera ($I_0 = 0$). One group was the result of averaging 5 frames; the other, 10 frames.

Figure 2 shows the mean and the root of the variance obtained from the experimental data, represented by the open circles. The solid curves were obtained from the evaluation of Eqs. (5) and (6). As can be seen, the agreement is good.

We begin with the definition of the discrete centroid to determine how the use of thresholding affects centroid computation:

$$X = \frac{\sum_j x_j I_j}{\sum_j I_j}\cdot \quad (7)$$

Now, considering that the recorded intensity over each pixel can be modeled with Eq. (1), we can express the centroid as

$$X = \frac{\sum_j x_j I_{0j}}{\sum_j (I_{0j} + n_j)} + \frac{\sum_j x_j n_j}{\sum_j (I_{0j} + n_j)}\cdot \quad (8)$$

Considering an n Gaussian with zero mean, we can extract the first moment of the centroid:

$$\langle X \rangle = \frac{\sum_j x_j I_{0j}}{\sum_j (I_{0j} + \langle n_j \rangle)} + \frac{\sum_j x_j \langle n_j \rangle}{\sum_j (I_{0j} + \langle n_j \rangle)}\cdot \quad (9)$$

Without thresholding, the average of the noise is null, so the mean centroid corresponds to the real centroid. This means that if we average over an infinite set of centroid samples we will obtain the correct intensity distribution. In that case, the mean is an unbiased estimate of the correct value of the centroid for the intensity profile.

However, as was shown above, the thresholding process induces the appearance of an effective noise

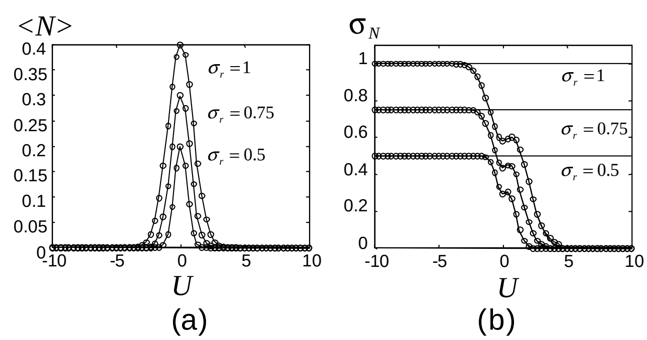

Fig. 1. (a) Mean and (b) variance of the effective noise for different thresholding levels with simulated data.

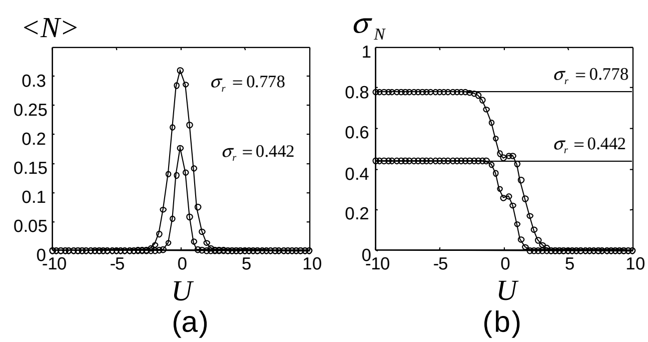

Fig. 2. (a) Mean and (b) variance of the effective noise for different thresholding levels with experimental data.

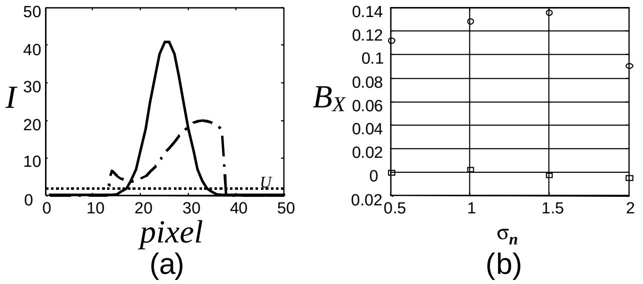

Fig. 3. (a) Simulated intensity profiles. (b) Bias of the centroid estimation in the presence of thresholding for different intensity profiles. ○, symmetric; □, asymmetric.

that has a nonnull mean. Moreover, this noise is not uniform because of its dependence on the difference between the intensity and the threshold over each pixel. If the intensity is circularly symmetrical, the mean centroid will be the correct centroid again. However, for symmetrical intensity profiles, it will not be a correct estimate of the real centroid (X_R); the mean centroid will be displaced because of the thresholding performed before computation of the centroid.

We designed a computer simulation to show the possible magnitude of this last effect. For two different intensity profiles [Fig. 3(a)] we computed the centroid error ($B_X = X_R - \langle X \rangle$) that is caused by thresholding of simulated data affected by different noise levels σ_n [see Fig. 3(b)].

We can observe from Fig. 3(b) that the value of the centroid for the asymmetrical intensity profile [dashed–dotted curve in Fig. 3(a)] shows a false displacement near 0.1 pixels. It is very important to point out that this bias cannot be attributed to spatial truncation of the intensity profile but is instead due to the effect that we have discussed here. As is expected, the bias obtained by simulation for the symmetrical Gaussian profile [solid curve in Fig. 3(a)] is very close to zero.

To complete the presentation of the statistical properties of the centroid we examine the variance, defined as

$$\sigma_X{}^2 = \langle X^2 \rangle - \langle X \rangle^2. \qquad (10)$$

When thresholding is not used, it is straightforward to find that the result of the variance corresponds to the case considered nowadays.[3] This variance does not depend on the intensity profile but on the total intensity. A complete study can be found in Ref. 3.

However, as we have found by testing, if we use thresholding, the mean value of the centroid can be different from the real value and, in this way, can affect the value of the variance of the centroid. Following this idea and regarding the centroid, we would obtain a different variance from the expected one if we considered the classical model of additive Gaussian noise of zero mean. We explain this difference as effective modification of the noise by preprocessing. Furthermore, the value of the variance depends on the symmetrical characteristics of the intensity profile.

In this Letter we have shown how the influence of thresholding of intensity data affected by additive Gaussian noise of zero mean can be attributed to effective noise, the mean and variance of which depend on the difference between the intensity and the threshold over each pixel. We also presented a theoretical model of the effective noise, accompanied by analytical expressions for the first and second moments of its probability distribution. The model was tested with experimental and simulated data. We have also discussed how thresholding preprocessing can affect the statistics of the calculated centroid. By computer simulation, we found that thresholding intensity data before the calculation of the centroid can induce a bias error in its estimation; this error was ~0.1 pixels for a particular asymmetrical intensity profile. This effect is more important for a greater loss of spatial symmetry in the intensity distribution and a smaller signal-to-noise ratio of the intensity profile.

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